A new function space and applications

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Abstract

We define a new function space B, which contains in particular BMO, BV, and $W^{1/p,p}$, $1 < p < \infty$. We investigate its embedding into Lebesgue and Marcinkiewicz spaces. We present several inequalities involving L^p norms of integer-valued functions in B. We introduce a significant closed subspace, B_0, of B, containing in particular VMO and $W^{1/p,p}$, $1 \le p < \infty$. The above mentioned estimates imply in particular that integer-valued functions belonging to B_0 are necessarily constant. This framework provides a "common roof" to various, seemingly unrelated, statements asserting that integer-valued functions satisfying some kind of regularity condition must be constant.

1 Introduction

Let Ω be a connected domain in $\mathbb{R}^n$. Recall that if $f : \Omega \to \mathbb{Z}$ is a measurable function which satisfies one of the following regularity properties:

1. $f \in \text{VMO}\,(\Omega)$;
2. $f \in W^{1,1}(\Omega)$;
3. $f \in W^{1/p,p}(\Omega)$, with $1 < p < \infty$,

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then f is constant [3, Comment 2, pp. 223-224], [2, Theorem B.1]. The original motivation for this article was to provide a "common roof" to all these cases, and which yields in particular the following

Theorem 1. *Assume that $f : \Omega \to \mathbb{Z}$ is measurable and can be written as $f = f_1 + f_2 + f_3$, with $f_1 \in \mathrm{VMO}\,(\Omega;\mathbb{R})$, $f_2 \in W^{1,1}(\Omega;\mathbb{R})$ and $f_3 \in W^{1/p,p}(\Omega;\mathbb{R})$ for some $1 < p < \infty$. Then f is constant.*

The proof of Theorem 1 relies heavily on the introduction of a new space of functions, which might be of interest well beyond the scope of Theorem 1.

In what follows we denote by Q the unit cube $(0,1)^n$. For $0 < \varepsilon < 1$, $Q_\varepsilon(a)$ is the ε-cube centered at a.

Given $f \in L^1(Q;\mathbb{R})$ and an ε-cube $Q_\varepsilon \subset Q$, we set

$$M(f,Q_\varepsilon) = \fint_{Q_\varepsilon} |f - f_{Q_\varepsilon}|, \text{ where } f_{Q_\varepsilon} = \fint_{Q_\varepsilon} f, \tag{1}$$

and

$$M^*(f,Q_\varepsilon) = \fint_{Q_\varepsilon}\fint_{Q_\varepsilon} |f(y) - f(z)|\, dy dz. \tag{2}$$

Clearly, we have

$$M(f,Q_\varepsilon) \le M^*(f,Q_\varepsilon) \le 2M(f,Q_\varepsilon). \tag{3}$$

Note that if $f = \mathbb{1}_A$, with $A \subset Q$ measurable, then

$$M(f,Q_\varepsilon) = M^*(f,Q_\varepsilon) = \frac{2|A\cap Q_\varepsilon|(|Q_\varepsilon| - |A\cap Q_\varepsilon|)}{|Q_\varepsilon|^2} \le \frac{1}{2}. \tag{4}$$

The following quantity plays an important role:

$$[f]_\varepsilon = \sup_{\mathscr{F}} \left\{ \varepsilon^{n-1} \sum_{j\in J} M(f, Q_\varepsilon(a_j)) \right\}. \tag{5}$$

Here, $\mathscr{F}$ denotes a collection of mutually disjoint ε-cubes, $\mathscr{F} = (Q_\varepsilon(a_j))_{j\in J}$, such that $\#J$ = cardinality of $J \le 1/\varepsilon^{n-1}$ (instead of $\#J$ we sometimes write $\#\mathscr{F}$) and the sup in (5) is taken over all such collections.

We then introduce the space

$$B = \left\{ f \in L^1(Q;\mathbb{R});\ \sup_{0<\varepsilon<1} [f]_\varepsilon < \infty \right\},$$

and the corresponding norm (modulo constants)

$$\|f\|_B = \sup_{0<\varepsilon<1} [f]_\varepsilon. \tag{6}$$

The definition of B is inspired by the celebrated BMO space of John–Nirenberg [4] equipped with the norm (modulo constants)

$$\|f\|_{\mathrm{BMO}} := \sup_{0<\varepsilon<1} \sup_{a\in Q} \{M(f,Q_\varepsilon(a)); Q_\varepsilon(a)\subset Q\}. \tag{7}$$

Here are several examples of functions in B.

Example 1. BMO $\subset B$ with continuous injection.
Indeed, using (7) we find that $\|f\|_B \le \|f\|_{\mathrm{BMO}}$.

When $n=1$, we clearly have $B=$ BMO ; however, when $n\ge 2$, B is strictly bigger than BMO (see e.g. Example 2 below).

Example 2. BV $\subset B$ with continuous injection.
Indeed, by Poincaré's inequality

$$\fint_{Q_\varepsilon} |f-f_{Q_\varepsilon}| \le \frac{c_n}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\nabla f|,$$

so that

$$\sum_{j\in J} M(f,Q_\varepsilon(a_j)) \le \frac{c_n}{\varepsilon^{n-1}} \int_{\cup_{j\in J} Q_\varepsilon(a_j)} |\nabla f| \tag{8}$$

and

$$[f]_\varepsilon \le c_n \int_Q |\nabla f|. \tag{9}$$

Example 3. $W^{1/p,p}\subset B$, $1<p<\infty$, with continuous injection.
Indeed, for every fixed $\alpha>0$ we have

$$\int_{Q_\varepsilon}\int_{Q_\varepsilon} |f(y)-f(z)|\,dydz \le n^{\alpha/2}\varepsilon^\alpha \int_{Q_\varepsilon}\int_{Q_\varepsilon} \frac{|f(y)-f(z)|}{|y-z|^\alpha}\,dydz.$$

Choosing $\alpha=(n+1)/p$ and applying Hölder's inequality gives

$$M^*(f,Q_\varepsilon) \le \frac{c_n}{\varepsilon^{(n-1)/p}} \left[\int_{Q_\varepsilon}\int_{Q_\varepsilon} \frac{|f(y)-f(z)|^p}{|y-z|^{n+1}}\,dydz\right]^{1/p}, \text{ with } c_n = n^{(n+1)/2},$$

and since $\#J \le 1/\varepsilon^{n-1}$ we obtain

$$\varepsilon^{n-1}\sum_{j\in J} M^*(f,Q_\varepsilon(a_j)) \le c_n \left[\sum_{j\in J}\int_{Q_\varepsilon(a_j)}\int_{Q_\varepsilon(a_j)} \frac{|f(y)-f(z)|^p}{|y-z|^{n+1}}\,dydz\right]^{1/p}. \tag{10}$$

Therefore

$$[f]_\varepsilon \le c_n \|f\|_{W^{1/p,p}}.$$

An important quantity associated with B is defined by

$$[f] = \overline{\lim_{\varepsilon\to 0}}\, [f]_\varepsilon. \tag{11}$$

The subspace

$$B_0 = \{f \in B; [f] = 0\} \tag{12}$$

plays a key role in this article.

Example 1'. VMO $\subset B_0$.
This is clear, since VMO functions (see [5]) are characterized by

$$\lim_{\varepsilon\to 0} \sup_{a\in Q} \{M(f, Q_\varepsilon(a)); Q_\varepsilon(a) \subset Q\} = 0.$$

Moreover, VMO $= B_0$ when $n = 1$.

Example 2'. $W^{1,1} \subset B_0$.
This is clear from (8) and the fact that $\left|\cup_{j\in J} Q_\varepsilon(a_j)\right| \le \varepsilon$.

Example 3'. $W^{1/p,p} \subset B_0$, $1 < p < \infty$.
This is an immediate consequence of (10) and the fact that $\left|\cup_{j\in J} Q_\varepsilon(a_j) \times Q_\varepsilon(a_j)\right| \le \varepsilon^{n+1}$.

In particular we see that

$$\text{VMO} + W^{1,1} + W^{1/p,p} \subset B_0. \tag{13}$$

2 Some properties of B

The main result of this section is

Theorem 2. *Let $n \ge 2$. Then we have $B \subset L^{n/(n-1),w}$, and*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1),w}} \le C_n \|f\|_B, \ \forall f \in B. \tag{14}$$

In Theorem 2, the Marcinkiewicz space $L^{n/(n-1),w}$ cannot be replaced by $L^{n/(n-1)}$, as a consequence of the next result.

Proposition 3. *Let $n \ge 2$. There exists some $f \in B$ such that $f \notin L^{n/(n-1)}$.*

Proof of Theorem 2. We may assume that

$$\|f\|_B \le 1 \text{ and } \fint_Q f = 0. \tag{15}$$

We also temporarily make the additional assumption that $f \in L^\infty$.

Under these assumptions, we will prove that

$$\|f\|_{L^{n/(n-1),w}} \simeq \sup_{t>0} t\,|\{|f|>t\}|^{(n-1)/n} \le C_n. \tag{16}$$

For this purpose it suffices to consider, in (16), only $t \gtrsim 1$.
We first note that, by (15), we have

$$\int_Q |f| \le 1. \tag{17}$$

In view of (17) we may consider, for $t > 1$, a Calderón-Zygmund decomposition at height t, i.e., we consider families $\mathscr{F}_j$ (with $j \ge 1$) of mutually disjoint cubes $Q_{2^{-j}} \subset Q$ of size 2^{-j} such that, if we set $\mathscr{F} = \cup_{j\ge 1}\mathscr{F}_j$, then

$$⨍_{Q_*} |f| \simeq t \text{ for every } Q_* \in \mathscr{F} \tag{18}$$

and

$$|f| \le t \text{ a.e. in } R := Q \setminus \cup_{Q_*\in\mathscr{F}} Q_*. \tag{19}$$

We next decompose $f = g + h$, with

$$g = f\mathbb{1}_R + \sum_{Q_*\in\mathscr{F}} \left(⨍_{Q_*} f\right)\mathbb{1}_{Q_*},$$

$$h = \sum_{j\ge 1} h_j, \text{ and } h_j = \sum_{Q_*\in\mathscr{F}_j} \left(f - ⨍_{Q_*} f\right)\mathbb{1}_{Q_*}.$$

By (18) and (19), we have

$$|g| \le Ct \text{ and thus } \{|f| > 2Ct\} \subset \{|h| > Ct\}. \tag{20}$$

Using (20), we see that (16) amounts to the following:

$$\sup_{t>1} t\,|\{|h| > Ct\}|^{(n-1)/n} \le c. \tag{21}$$

We now proceed with the proof of (21). Since $\|f\|_B = 1$, for every family $\mathscr{G} \subset \mathscr{F}_j$ such that

$$\#\mathscr{G} \le 1/(2^{-j})^{n-1} = 2^{j(n-1)},$$

we have

$$2^{-j(n-1)} \sum_{Q_*\in\mathscr{G}} ⨍_{Q_*} \left| f - ⨍_{Q_*} f \right| \le 1.$$

By covering $\mathscr{F}_j$ with mutually disjoint sets $\mathscr{G}$ as above, we find that

$$\sum_{Q_*\in\mathscr{F}_j} ⨍_{Q_*} \left| f - ⨍_{Q_*} f \right| \le 2^{j(n-1)} + \#\mathscr{F}_j, \tag{22}$$

and thus

$$\|h_j\|_{L^1} \leq 2^{-j} + 2^{-nj} \# \mathscr{F}_j. \tag{23}$$

On the other hand, we have (using (18))

$$1 \geq \|f\|_{L^1} \geq \sum_{j \geq 1} \sum_{Q_* \in \mathscr{F}_j} \int_{Q_*} |f| = \sum_{j \geq 1} \sum_{Q_* \in \mathscr{F}_j} 2^{-nj} ⨍_{Q_*} |f| \gtrsim \sum_{j \geq 1} 2^{-nj}\, t\, \# \mathscr{F}_j. \tag{24}$$

From (23) and (24), we deduce that

$$\sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \sum_{\mathscr{F}_j \neq \emptyset} 2^{-j}. \tag{25}$$

We next recall that

$$\|f\|_{L^{n/(n-1),w}} = \sup_{A \subset Q} |A|^{-1/n} \int_A |f|. \tag{26}$$

If $\mathscr{F}_j \neq \emptyset$ and $Q_* \in \mathscr{F}_j$, then (26) applied with $A = Q_*$, combined with (18), implies that

$$2^{-j} \lesssim \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \tag{27}$$

By (25) and (27), we have

$$\|h\|_{L^1} \leq \sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t} \right)^{1/(n-1)}. \tag{28}$$

In turn, (28) implies that (with C as in (21))

$$|\{|h| > Ct\}| \leq \frac{\|h\|_{L^1}}{Ct} \lesssim \frac{1}{t^2} + \left(\frac{\|f\|_{L^{n/(n-1),w}}}{t^n} \right)^{1/(n-1)}, \tag{29}$$

and thus

$$t\, |\{|h| > Ct\}|^{(n-1)/n} \lesssim t^{(2-n)/n} + \|f\|_{L^{n/(n-1),w}}^{1/n} \leq 1 + \|f\|_{L^{n/(n-1),w}}^{1/n}. \tag{30}$$

By taking, in (30), the supremum over $t > 1$, we find that

$$\|f\|_{L^{n/(n-1),w}} \lesssim 1 + \|f\|_{L^{n/(n-1),w}}^{1/n},$$

and therefore $\|f\|_{L^{n/(n-1),w}} \lesssim 1$.

We complete the proof by removing the assumption that $f \in L^\infty$. Let

$$\Phi_N(s) = \begin{cases} s, & \text{if } |s| \leq N \\ N, & \text{if } s > N \\ -N, & \text{if } s < -N \end{cases}$$

and set $f_N := \Phi_N(f)$. By (3), we have $\|f_N\|_B \le 2\|f\|_B$. In addition, f_N is bounded and thus satisfies (14), i.e.,

$$\left\| f_N - \fint_Q f_N \right\|_{L^{n/(n-1),w}} \le 2C_n \|f\|_B. \tag{31}$$

Using (26) and passing to the limit as $N \to \infty$ in (31) yields (14) for every $f \in B$. □

Proof of Proposition 3. Set

$$\varphi(x) = (1 - |x|)^+, \quad \forall\, x \in \mathbb{R}^n$$

and

$$N_m = 2^{2^m}, \quad \forall\, m \ge 1.$$

Consider a sequence of points $(b_m)_{m\ge1}$ such that the open balls $B(b_m, 2/N_m)$ are contained in Q and mutually disjoint. (We may e.g. choose the points b_m on a line segment parallel to the x_1-axis.) Set

$$f_m(x) = N_m^{n-1}\varphi(N_m(x - b_m)), \quad \forall\, m \ge 1 \tag{32}$$

and

$$f(x) = \sum_{m\ge1} f_m(x). \tag{33}$$

We will prove that $f \in B$ and $f \notin L^{n/(n-1)}$.

Note that

$$\operatorname{supp} f_m = \overline{B}(b_m, 1/N_m),$$

and that the sets $\operatorname{supp} f_m$, $m \ge 1$, are mutually disjoint.

Clearly,

$$\|f_m\|_{L^1(Q)} = \frac{C}{N_m}, \quad \forall\, m \ge 1, \tag{34}$$

and thus $f \in L^1(Q)$; here and in what follows we denote by C a generic constant depending only on n,

We have

$$\|f_m\|^{n/(n-1)}_{L^{n/(n-1)}(Q)} = C, \quad \forall\, m \ge 1,$$

so that $f \notin L^{n/(n-1)}(Q)$.

Given $0 < \varepsilon < 1$ and integers $M_1 = M_1(\varepsilon) \ge 1$ and $M_2 = M_2(\varepsilon) > M_1(\varepsilon)$ to be defined later, we write

$$f = S_1 + S_2 + S_3, \tag{35}$$

with

$$S_1 = \sum_{m \le M_1} f_m, \quad S_2 = \sum_{M_1 < m \le M_2} f_m, \quad S_3 = \sum_{m > M_2} f_m. \tag{36}$$

We now estimate separately $[S_1]_\varepsilon$, $[S_2]_\varepsilon$ and $[S_3]_\varepsilon$.

Estimate of $[S_1]_\varepsilon$. Here we use the fact that if $h \in \mathrm{Lip}(Q)$ then

$$M(h, Q_\varepsilon(a)) \le \sqrt{n}\,\varepsilon \,\|h\|_{\mathrm{Lip}}, \tag{37}$$

and thus

$$[h]_\varepsilon \le \sqrt{n}\,\varepsilon \,\|h\|_{\mathrm{Lip}}.$$

In particular,

$$[f_m]_\varepsilon \le C\,\varepsilon (N_m)^n. \tag{38}$$

Using (38) and the fact that

$$\sum_{i=1}^{p} X^i \le \frac{X^{p+1}}{X-1}, \quad \forall X > 1,$$

we deduce that

$$[S_1]_\varepsilon \le C\,\varepsilon\, 2^{n\,2^{M_1}}, \quad \forall \varepsilon \in (0,1). \tag{39}$$

Estimate of $[S_2]_\varepsilon$. Applying (9) to f_m yields

$$[f_m]_\varepsilon \le C, \quad \forall m \ge 1, \ \forall \varepsilon \in (0,1),$$

and in particular

$$[S_2]_\varepsilon \le C(M_2 - M_1), \quad \forall \varepsilon \in (0,1). \tag{40}$$

Estimate of $[S_3]_\varepsilon$. Clearly

$$[h]_\varepsilon \le \frac{2}{\varepsilon} \|h\|_{L^1(Q)}, \quad \forall h \in L^1. \tag{41}$$

From (34) we deduce that

$$[f_m]_\varepsilon \le \frac{C}{\varepsilon N_m}. \tag{42}$$

Using (42) and the fact that

$$\sum_{i=p}^{\infty} Y^i = \frac{Y^p}{1-Y}, \quad \forall Y \in [0,1),$$

we see that

$$[S_3]_\varepsilon \le \frac{C}{\varepsilon\, 2^{2^{M_2}}}. \tag{43}$$

We now explain how to choose $M_1(\varepsilon)$ and $M_2(\varepsilon)$. Given $0<\varepsilon<1$, we denote by $M_1 = M_1(\varepsilon)$ the largest integer $\ell \ge 1$ such that

$$2^{n\,2^\ell} \le \frac{2^{2n}}{\varepsilon}. \tag{44}$$

Equivalently, we have

$$2^{n\,2^{M_1}} \le \frac{2^{2n}}{\varepsilon} \tag{45}$$

and

$$2^{2n\,2^{M_1}} > \frac{2^{2n}}{\varepsilon}. \tag{46}$$

Combining (39) and (45) yields

$$[S_1]_\varepsilon \le C, \quad \forall\, \varepsilon \in (0,1). \tag{47}$$

From (45) and (46) we obtain

$$|M_1(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \le C, \quad \forall\, \varepsilon \in (0,1/2). \tag{48}$$

Next we denote by $M_2 = M_2(\varepsilon)$ the smallest integer $\ell \ge 1$ such that

$$2^{2^\ell} \ge \frac{4}{\varepsilon}.$$

(Note that $M_2 > M_1$ since $2^{2^{M_1}} < 4/\varepsilon$.)
Equivalently, we have

$$2^{2^{M_2}} \ge \frac{4}{\varepsilon} \tag{49}$$

and

$$2^{2^{M_2-1}} < \frac{4}{\varepsilon}. \tag{50}$$

Combining (43) and (49) yields

$$[S_3]_\varepsilon \le C, \quad \forall\, \varepsilon \in (0,1). \tag{51}$$

From (49) and (50) we obtain

$$|M_2(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \le C, \quad \forall\, \varepsilon \in (0,1/2). \tag{52}$$

Therefore,

$$|M_2(\varepsilon) - M_1(\varepsilon)| \le C, \quad \forall \varepsilon \in (0,1). \tag{53}$$

(Inequality (53) is deduced from (48) and (52) when $\varepsilon \in (0,1/2)$, and from (50) when $\varepsilon \in [1/2,1)$.)

It follows from (40) and (53) that

$$[S_2]_\varepsilon \le C, \quad \forall \varepsilon \in (0,1). \tag{54}$$

Putting together (47), (51) and (54) we conclude that

$$[f]_\varepsilon \le C, \quad \forall \varepsilon \in (0,1),$$

and thus $f \in B$. □

3 Some properties of B_0 and $[f]$

Our first result is

Theorem 4. *Let f be a $\mathbb{Z}$-valued function on Q such that $f \in B_0$. Then f is constant.*

Combining Theorem 4 with (13) we obtain Theorem 1.

When $n = 1$ we have $B_0 = \mathrm{VMO}$ and we may then invoke the fact that functions in VMO $(Q;\mathbb{Z})$ are constant (for any $n \ge 1$); see [3, Comment 2, p. 223–224]. Therefore it suffices to prove Theorem 4 when $n \ge 2$. Next, we observe that it suffices to prove Theorem 4 when $f = \mathbb{1}_A$ for some $A \subset Q$. Indeed, let $k \in \mathbb{Z}$ be such that $|f^{-1}(k)| > 0$. Set $A = f^{-1}(k)$ and $g = \mathbb{1}_A$. Clearly $M^*(f,Q_\varepsilon) \ge M^*(g,Q_\varepsilon)$ for every ε-cube Q_ε. Since $f \in B_0$, we deduce that $g \in B_0$. If Theorem 4 holds for g, then $g \equiv 1$, and thus $f \equiv k$.

Hence it remains to prove Theorem 4 when $n \ge 2$ and $f = \mathbb{1}_A$. In this case we have the following quantitative improvement of Theorem 4.

Theorem 5. *Let $n \ge 2$. There exists a constant C_n such that if $f = \mathbb{1}_A$ with $A \subset Q$ measurable, then*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \le C_n [f]. \tag{55}$$

Remark 6. A much more precise result (see [1]) asserts that there exist two constants $0 < \underline{c}_n \le \overline{c}_n < \infty$ such that if $f = \mathbb{1}_A$, then

$$\underline{c}_n \min\left\{1, \int_Q |\nabla f|\right\} \le [f] \le \overline{c}_n \min\left\{1, \int_Q |\nabla f|\right\}, \tag{56}$$

with the convention that $\int_Q |\nabla f| = \infty$ if $f \notin \mathrm{BV}$.

Note that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \int_Q |\nabla f| \tag{57}$$

by the Sobolev embedding, and that clearly

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \ \text{ when } f = \mathbb{1}_A. \tag{58}$$

Therefore

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \min\left\{1, \int_Q |\nabla f|\right\} \leq C'[f] \ \text{ by (56).}$$

In fact, using a variant of the definition (5) involving ε-cubes of general orientation, one obtains a quantity $[f]_\varepsilon^*$ satisfying

$$[f]_\varepsilon \leq [f]_\varepsilon^* \leq C_1 [f]_{C_2\varepsilon}$$

for some constants $C_1 > 1$, $C_2 > 1$ depending only on n (see [1]). The main result in [1] asserts that if $f = \mathbb{1}_A$, then

$$\lim_{\varepsilon \to 0} [f]_\varepsilon^* = \frac{1}{2} \min\left\{1, \int_Q |\nabla f|\right\}; \tag{59}$$

the ingredients of the proof of (59) are much more sophisticated than the arguments presented below. We acknowledge that it was Theorem 5 which prompted one of us to conjecture that (59) holds.

The main tool in the proof of Theorem 5 is

Lemma 7. *Let $n \geq 2$. Let $U = \cup_{j \in J} Q_\varepsilon(a_j)$ be a union of ε-cubes. Then $Q \setminus U$ contains a connected set S of measure $\geq 1 - \alpha_n (\#J)^{n/(n-1)} \varepsilon^n$, for some positive constant α_n depending only on n.*

Here, the ε-cubes are not necessarily mutually disjoint, and we do not assume that these cubes are completely contained in Q.

Remark 8. The conclusion of Lemma 7 is optimal. Indeed, consider a ball $B \subset Q$ of (small) radius R. We may cover the sphere $\Sigma = \partial B$ by a union of ε-cubes as above with $\#J\,\varepsilon^{n-1} \simeq R^{n-1}$. Then $|B| \simeq R^n \simeq (\#J)^{n/(n-1)} \varepsilon^n$.

Granted Lemma 7, we turn to the

Proof of Theorem 5. Let $f = \mathbb{1}_A$, with $A \subset Q$. *Fix* any $\lambda \in (0, 1/2)$, e.g. $\lambda = 1/4$.

In view of (58), we may assume that

$$0 \leq [f] < 2\lambda(1-\lambda), \tag{60}$$

for otherwise the conclusion is clear with $C_n = \dfrac{1}{\lambda(1-\lambda)}$.

Note that, by (4),

$$M(f,Q_\varepsilon) = 2f_{Q_\varepsilon}(1-f_{Q_\varepsilon}).$$

Therefore,

$$M(f,Q_\varepsilon) < 2\lambda(1-\lambda) \implies \text{either } f_{Q_\varepsilon} < \lambda, \text{ or } f_{Q_\varepsilon} > 1-\lambda. \tag{61}$$

With ε small and $\widetilde{Q} = (\varepsilon, 1-\varepsilon)^n$, consider a maximal family $J = J_\varepsilon$ of points $a \in \widetilde{Q}$ such that the cubes $Q_\varepsilon(a)$ are mutually disjoint and satisfy

$$M(f,Q_\varepsilon(a)) \geq 2\lambda(1-\lambda),\ \forall\, a \in J. \tag{62}$$

Let $\nu > 0$ (to be chosen arbitrarily small later). We claim that for ε sufficiently small (depending on ν) we have

$$\#J \leq \delta/\varepsilon^{n-1},\ \ \text{with } \delta = \frac{[f]+\nu}{2\lambda(1-\lambda)}. \tag{63}$$

Indeed, we first see that, for ε sufficiently small,

$$\#J \leq 1/\varepsilon^{n-1}. \tag{64}$$

Otherwise, we may choose a subfamily $\widetilde{J}$ such that $\#\widetilde{J} = I(1/\varepsilon^{n-1})$, where $I(t)$ denotes the integer part of t. Then

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#\widetilde{J})\, 2\lambda(1-\lambda) \geq \varepsilon^{n-1}\left(\frac{1}{\varepsilon^{n-1}} - 1\right) 2\lambda(1-\lambda),$$

which, by (60), is impossible for ε small. From (64) and the definition of $[f]_\varepsilon$ we have

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#J)\, 2\lambda(1-\lambda),$$

which yields (63) for ε sufficiently small.

Set $U := \cup_{a\in J} Q_{2\varepsilon}(a)$. By Lemma 7 and a scaling argument, $\widetilde{Q} \setminus U$ contains a connected set $S = S_\varepsilon$ such that

$$|S_\varepsilon| \geq (1-2\varepsilon)^n - \alpha'_n\, \delta^{n/(n-1)}, \tag{65}$$

where $\alpha'_n = 2^n\, \alpha_n$.

We next note that (by the maximality of J) U contains the set

$$V = V_\varepsilon := \{b \in \widetilde{Q};\, M(f,Q_\varepsilon(b)) \geq 2\lambda(1-\lambda)\}, \tag{66}$$

and thus $S \subset \widetilde{Q} \setminus V$.

We consider the continuous function

$$f_\varepsilon : \widetilde{Q} \to \mathbb{R},\ f_\varepsilon(a) = f_{Q_\varepsilon(a)}.$$

By (61) and (66), in the set $\widetilde{Q}\setminus V$ the function f_ε takes values into $[0,\lambda)\cup(1-\lambda,1]$. $S\subset\widetilde{Q}\setminus V$ being connected, we find that either $f_\varepsilon<\lambda$, or $f_\varepsilon>1-\lambda$ in S.

We assume e.g. that $f_\varepsilon<\lambda$ in S_ε along a sequence $\varepsilon_m\to 0$. Clearly,

$$\int_{A\cap\widetilde{Q}}|1-f_\varepsilon|\to 0 \ \text{ as } \varepsilon\to 0,$$

and thus

$$(1-\lambda)\left|S_{\varepsilon_m}\cap A\right|\le\int_{S_{\varepsilon_m}\cap A}(1-f_{\varepsilon_m})\to 0 \ \text{ as } m\to\infty. \tag{67}$$

On the other hand, by (65) and (67) we have

$$|A|=\left|S_{\varepsilon_m}\cap A\right|+\left|\left(\widetilde{Q}\setminus S_{\varepsilon_m}\right)\cap A\right|+\left|\left(Q\setminus\widetilde{Q}\right)\cap A\right|\le\alpha'_n\,\delta^{n/(n-1)}+o(1) \ \text{ as } m\to\infty,$$

and thus $|A|\le\alpha'_n\,\delta^{n/(n-1)}$, so that

$$|A|^{(n-1)/n}\le\alpha''_n\,\delta=\alpha''_n\,\frac{[f]+\nu}{2\lambda(1-\lambda)}, \ \text{ with } \alpha''_n=\left(\alpha'_n\right)^{(n-1)/n}.$$

Since $\nu>0$ can be chosen arbitrarily small, we deduce that

$$|A|^{(n-1)/n}\le\frac{\alpha''_n[f]}{2\lambda(1-\lambda)}. \tag{68}$$

Finally, we note that

$$\begin{aligned}\left\|f-\fint f\right\|_{L^{n/(n-1)}}&=\left(|A|(1-|A|)^{n/(n-1)}+(1-|A|)|A|^{n/(n-1)}\right)^{(n-1)/n}\\&\le 2\min\left\{|A|^{(n-1)/n},\,|A^c|^{(n-1)/n}\right\}.\end{aligned} \tag{69}$$

Combining (68) and (69) yields (55). □

For further use, let us note that the proof of Theorem 5 leads to the following result.

Lemma 9. *Let $n\ge 2$ and $\lambda\in(0,1/2)$. Let $A\subset Q$ be measurable and set $f:=\mathbb{1}_A$. Assume that there exists a sequence $\varepsilon_m\to 0$ and families*

$$J_m\subset\widetilde{Q}^m:=(3\varepsilon_m,1-3\varepsilon_m)^n$$

of points a with the following property:
If $b\in\widetilde{Q}^m\setminus\cup_{a\in J_m}Q_{2\varepsilon_m}(a)$, then $M(f,Q_{\varepsilon_m}(b))<2\lambda(1-\lambda)$.
Let

$$\delta:=\varliminf_{m\to\infty}\varepsilon_m^{n-1}\#J_m.$$

Then either $|A|\ge 1-\widetilde{c}_n\delta^{n/(n-1)}$, or $|A^c|\ge 1-\widetilde{c}_n\delta^{n/(n-1)}$.

Proof of Lemma 7. Recall a standard "relative" isoperimetric inequality. Let $B \subset Q$ satisfy $|B| \le 1/2$. By (57) (applied with $f = \mathbb{1}_B$) and (69), we have

$$|B| \le c_n \left(\int_Q |\nabla \mathbb{1}_B| \right)^{n/(n-1)} = c_n [P(B)]^{n/(n-1)}, \tag{70}$$

where $P(B)$ represents the perimeter of B relative to Q. When B is a Lipschitz domain (which will be the case in what follows), $P(B)$ is the (surface) measure of $\partial B \cap Q$.

We now turn to the proof of the lemma. Set $\delta = (\# J)\varepsilon^{n-1}$. Let $(A_i)_{i\in I}$ be the connected components of the open set $Q \setminus \cup_{j\in J} \overline{Q}_\varepsilon(a_j)$.

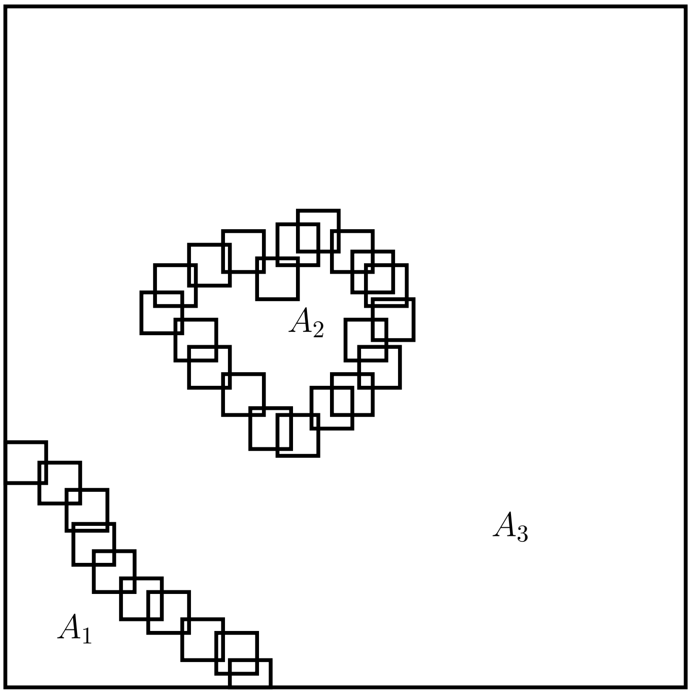

Figure 1: The components of $Q \setminus \cup_{j\in J} \overline{Q}_\varepsilon(a_j)$.

Note that each A_i is Lipschitz, and that

$$\cup_{i\in I} (\partial A_i \cap Q) \subset \cup_{j\in J} (\partial Q_\varepsilon(a_j) \cap Q). \tag{71}$$

Let

$$G_j := \{x \in \partial Q_\varepsilon(a_j) \cap Q; x \text{ does not belong to the } (n-2) \text{ skeleton of } \partial Q_\varepsilon(a_j)\}.$$

Note that

$$[\cup_{i\in I} (\partial A_i \cap Q)] \setminus \left[\cup_{j\in J} G_j\right] \text{ has zero } \mathscr{H}^{n-1} - \text{measure.}$$

Since a point $x \in G_j$ belongs to at most one ∂A_i, we find, using (71), that

$$\sum_{i\in I} P(A_i) \le \sum_{j\in J} P(Q_\varepsilon(a_j)) \le c'_n \delta. \tag{72}$$

We claim that if $\delta < \delta_n$ (a positive number depending only on n), then there exists some $i_0 \in I$ such that $|A_{i_0}| > 1/2$. Indeed, argue by contradiction and assume that $|A_i| \le 1/2$, $\forall\, i \in I$. By (70) and (72), we have

$$\begin{aligned} 1-|U| = |Q\setminus U| = \sum_{i\in I}|A_i| \le c_n \sum_{i\in I}[P(A_i)]^{n/(n-1)} \le c_n \left[\sum_{i\in I} P(A_i)\right]^{n/(n-1)} \\ \le c_n(c'_n\delta)^{n/(n-1)} = c''_n \delta^{n/(n-1)}. \end{aligned} \tag{73}$$

On the other hand

$$|U| \le (\#J)\varepsilon^n = \delta\varepsilon < \delta. \tag{74}$$

Combining (73) and (74) we obtain

$$1 \le \delta + c''_n \delta^{n/(n-1)};$$

this is impossible when $\delta < \delta_n$, where δ_n is the solution of

$$1 = \delta_n + c''_n(\delta_n)^{n/(n-1)},$$

and thus the claim is established when $\delta < \delta_n$.

Set $S = A_{i_0}$, which is clearly connected and contained in $Q\setminus U$. Applying (70) to $B = S^c$ we find (using (72))

$$1-|S| \le c_n[P(S^c)]^{n/(n-1)} = c_n[P(S)]^{n/(n-1)} \le c''_n \delta^{n/(n-1)},$$

which is the desired conclusion when $\delta < \delta_n$.

Finally, we observe that

$$1 - \frac{1}{(\delta_n)^{n/(n-1)}}\delta^{n/(n-1)} \le 0$$

when $\delta \ge \delta_n$ and therefore Lemma 7 holds with

$$\alpha_n = \max\left\{c''_n, \frac{1}{(\delta_n)^{n/(n-1)}}\right\}. \qquad \square$$

4 An extension of Theorem 5 to $\mathbb{Z}$-valued functions

Our main result in this section is

Theorem 10. *Let $n \geq 2$. There exists a positive constant c (independent of n) such that if f is a $\mathbb{Z}$-valued function in B and $[f] < c$, then $f \in L^{n/(n-1)}(Q)$ and*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C_n [f], \tag{75}$$

for some constant C_n depending only on n.

Theorem 5 can be deduced from Theorem 10. Indeed, let $f = \mathbb{1}_A$. Then either $[f] \leq c$, and Theorem 10 applies, or $[f] > c$, and then

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \leq (2/c)[f].$$

The smallness condition on $[f]$ in Theorem 10 is essential, as shown by the following improvement of Proposition 3.

Proposition 11. *Let $n \geq 2$. There exists a $\mathbb{Z}$-valued function $f \in B$ such that $f \notin L^{n/(n-1)}(Q)$.*

Proof of Theorem 10. Step 1. Decomposition of f as a sum of characteristic functions.
We temporarily assume that $f \geq 0$. Then f is a sum of characteristic functions. Indeed, set

$$A_k := \{x \in Q; f(x) \geq k\}, \ \forall k \in \mathbb{N}^*,$$

and let $g_k := \mathbb{1}_{A_k}$. Then we claim that

$$f = \sum_{k>0} g_k \tag{76}$$

and

$$|f(x) - f(y)| = \sum_{k>0} |g_k(x) - g_k(y)|, \ \forall x, y \in Q. \tag{77}$$

Indeed, on the one hand (76) follows from

$$\sum_{k>0} g_k(x) = \sum_{0<k\leq f(x)} 1 = f(x).$$

On the other hand, assuming e.g. that $f(x) \geq f(y)$, we have $g_k(x) = g_k(y)$ provided either $k \leq f(y)$ or $k > f(x)$, and thus

$$\sum_{k>0} |g_k(x) - g_k(y)| = \sum_{f(y)<k\leq f(x)} |g_k(x) - g_k(y)| = \sum_{f(y)<k\leq f(x)} 1 = f(x) - f(y) = |f(x) - f(y)|;$$

that is, (77) holds.

We next note that (77) implies

$$M^*(f, Q_\varepsilon) = \sum_{k>0} M^*(g_k, Q_\varepsilon), \tag{78}$$

and in particular

$$M(g_k, Q_\varepsilon) \leq M^*(f, Q_\varepsilon), \ \forall\, k > 0. \tag{79}$$

Step 2. Construction of maximal families of "bad" cubes.
Fix some $\lambda \in (0, 1/2)$ and consider a sequence $\varepsilon_m \to 0$. Let $\widetilde{Q}^m := (3\varepsilon_m, 1 - 3\varepsilon_m)^n$. Let J_m be a maximal family of points $a \in \widetilde{Q}^m$ such that the cubes $Q_{\varepsilon_m}(a)$, $a \in J_m$, are mutually disjoint and satisfy $M^*(f, Q_{\varepsilon_m}(a)) \geq 2\lambda(1-\lambda)$.

By the maximality of J_m and by (79), we have

$$M(g_k, Q_{\varepsilon_m}(b)) \leq M^*(f, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \ \forall\, b \in \widetilde{Q}_m \setminus \cup_{a\in J_m} Q_{2\varepsilon_m}(a). \tag{80}$$

We next associate to each k an appropriate subfamily extracted from J_m. More specifically, let

$$J_m^k := \{a \in J_m;\, 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) \geq 2\lambda(1-\lambda)\}. \tag{81}$$

We claim that

$$M(g_k, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \ \forall\, b \in \widetilde{Q}_m \setminus \cup_{a\in J_m^k} Q_{2\varepsilon_m}(a). \tag{82}$$

Indeed, (80) implies that (82) holds for $b \in \widetilde{Q}_m \setminus \cup_{a\in J_m} Q_{2\varepsilon_m}(a)$.

It remains to establish (82) when $b \in Q_{2\varepsilon_m}(a)$ for some $a \in J_m \setminus J_m^k$. In this case, we have $Q_{\varepsilon_m}(b) \subset Q_{3\varepsilon_m}(a)$ and thus

$$M^*(g_k, Q_{\varepsilon_m}(b)) \leq 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) < 2\lambda(1-\lambda).$$

This completes the proof of (82).

Step 3. A first estimate of $\|f - \fint_Q f\|_{L^{n/(n-1)}}$.
By (69), (82), and Lemma 9, we have

$$\left\| g_k - \fint_Q g_k \right\|_{L^{n/(n-1)}} \leq 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \# J_m^k. \tag{83}$$

Thus

$$\sum_{k>0} \left\| g_k - \fint_Q g_k \right\|_{L^{n/(n-1)}} \leq 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k, \tag{84}$$

and therefore

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \le 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k. \tag{85}$$

Step 4. A second estimate of $\|f - \fint_Q f\|_{L^{n/(n-1)}}$.
In this step, we assume that

$$[f] < d := \lambda(1-\lambda), \quad \text{with } \lambda \text{ chosen as in Step 2.} \tag{86}$$

Under this assumption, we will prove that

$$c'_n \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k \le [f] \ \text{ for some constant } c'_n > 0. \tag{87}$$

Granted this estimate, we obtain (using (85)) that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \le \widetilde{C}_n [f], \quad \text{with } \widetilde{C}_n = 2(\widetilde{c}_n)^{(n-1)/n} / c'_n. \tag{88}$$

We now proceed to the proof of (87). We first note that (by (3)) we have

$$M(f, Q_{\varepsilon_m}(a)) \ge \lambda(1-\lambda), \ \forall a \in J_m. \tag{89}$$

Repeating the proof of (64) (and using (86) and (89)), for large m we have

$$\# J_m \le 1/(\varepsilon_m)^{n-1}. \tag{90}$$

We next rely on the following lemma, well-known to the experts, whose proof is omitted.

Lemma 12. *Let $\{Q_\varepsilon(a); a \in J\}$ be a family of mutually disjoint ε-cubes. Then there exists a constant $N = N(n)$ such that*

1. *$J = J^1 \cup J^2 \cup \ldots J^N$.*
2. *For every j, the cubes $Q_{3\varepsilon}(a)$, $a \in J^j$, are mutually disjoint.*
3. *For every j, we have $\# J^j \le \# J / 3^{n-1}$.*

By Lemma 12, for every family of mutually disjoint ε-cubes $Q_\varepsilon(a)$, $a \in J \subset (3\varepsilon, 1-3\varepsilon)^n$, such that $\# J \le 1/\varepsilon^{n-1}$, we have

$$(3\varepsilon)^{n-1} \sum_{a\in J} M(h, Q_{3\varepsilon}(a)) \le N [h]_{3\varepsilon}, \ \forall h : Q \to \mathbb{R}. \tag{91}$$

In particular, for large m we have (using (90) and (91))

$$(\varepsilon_m)^{n-1} \sum_{a\in J_m} M(f, Q_{3\varepsilon_m}(a)) \le N/3^{n-1} [f]_{3\varepsilon_m}. \tag{92}$$

Combining (92) with (3), we see that

$$(\varepsilon_m)^{n-1}\sum_{a\in J_m} M^*(f,Q_{3\varepsilon_m}(a)) \le 2N/3^{n-1}[f]_{3\varepsilon_m} \tag{93}$$

We now use successively (93), (78) and (81) and obtain that

$$\begin{aligned}
[f]_{3\varepsilon_m} &\ge 3^{n-1}/(2N)(\varepsilon_m)^{n-1}\sum_{a\in J_m} M^*(f,Q_{3\varepsilon_m}(a))\\
&= 3^{n-1}/(2N)(\varepsilon_m)^{n-1}\sum_{a\in J_m}\sum_{k>0} M^*(g_k,Q_{3\varepsilon_m}(a))\\
&\ge 3^{n-1}/(2N)(\varepsilon_m)^{n-1}\sum_{k>0}\sum_{a\in J_m^k} M^*(g_k,Q_{3\varepsilon_m}(a))\\
&\ge \lambda(1-\lambda)/(3^{n+1}N)(\varepsilon_m)^{n-1}\sum_{k>0}\#J_m^k = c_n'(\varepsilon_m)^{n-1}\sum_{k>0}\#J_m^k,
\end{aligned}\tag{94}$$

with $c_n' := \lambda(1-\lambda)/(3^{n+1}N)$.

We derive (87) by letting $m\to\infty$ in (94).

Step 5. We remove the assumption $f\ge 0$.
We note that $f = f^+ - f^-$, and that

$$|f^\pm(x)-f^\pm(y)| \le |f(x)-f(y)|,\ \forall x,y\in Q. \tag{95}$$

By (3) and (95), we have

$$M^*(f^\pm,Q_\varepsilon) \le M^*(f,Q_\varepsilon) \le 2M(f,Q_\varepsilon),$$

and thus $[f^\pm]\le 2[f]$. By the first part of the proof of this theorem, we have

$$\left\| f^\pm - \fint_Q f^\pm \right\|_{L^{n/(n-1)}} \le \widetilde{C}_n[f^\pm] \le 2\widetilde{C}_n[f], \tag{96}$$

provided $[f] < c := d/2$.

Finally, (96) implies that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \le C_n[f] \ \text{ provided } [f] < c,$$

with $C_n := 4\widetilde{C}_n$.

The proof of Theorem 10 is complete. □

Proof of Proposition 11. We use the same notation and the same strategy as in the proof of Proposition 3, with some minor modifications.

Set

$$g_m(x) = I(f_m(x)),\ \forall m\ge 1,\ \text{where } I(t) \text{ denotes the integer part of } t,$$

and

$$g(x) = \sum_{m\geq 1} g_m(x).$$

Clearly,

$$\|g_m\|_{L^1(Q)} \leq \|f_m\|_{L^1(Q)} = \frac{C}{N_m} \tag{97}$$

(by (34)), so that $g \in L^1(Q)$. On the other hand

$$\|g_m\|^{n/(n-1)}_{L^{n/(n-1)}(Q)} \geq \|f_m - 1\|^{n/(n-1)}_{L^{n/(n-1)}([f_m>1])} \geq \alpha > 0, \quad \forall m \geq 1,$$

and thus $g \notin L^{n/(n-1)}(Q)$.

We will now prove that $g \in B$. Write

$$g = T_1 + T_2 + T_3,$$

with

$$T_1 = \sum_{m\leq M_1} g_m, \quad T_2 = \sum_{M_1<m\leq M_2} g_m, \quad T_3 = \sum_{m>M_2} g_m,$$

where $M_1 = M_1(\varepsilon)$ and $M_2 = M_2(\varepsilon)$ are defined exactly as in the proof of Proposition 3.

Estimate of $[T_1]_\varepsilon$. Since $g_m \notin \mathrm{Lip}(Q)$, we need to modify the argument. We claim that, for sufficiently small ε (depending only on n), given any cube $Q_\varepsilon(a)$ there exists at most one integer $m \leq M_1(\varepsilon)$ such that

$$Q_\varepsilon(a) \cap (\operatorname{supp} g_m) \neq \emptyset. \tag{98}$$

Indeed, if (98) holds, then

$$Q_\varepsilon(a) \cap B(b_m, 1/N_m) \neq \emptyset,$$

and thus

$$Q_\varepsilon(a) \subset B(b_m, 2/N_m) \tag{99}$$

provided

$$\frac{1}{N_m} + \sqrt{n}\,\varepsilon \leq \frac{2}{N_m}, \quad \forall m \leq M_1. \tag{100}$$

On the other hand, (45) implies that

$$N_{M_1} \leq \frac{4}{\varepsilon^{1/n}},$$

and thus (100) holds when

$$\varepsilon \le \varepsilon_0 := \frac{1}{4^{n/(n-1)}\, n^{n/[2(n-1)]}}.$$

We deduce the claim using (99) and the fact that the balls $B(b_m, 2/N_m)$ are mutually disjoint.

Therefore, for $\varepsilon \le \varepsilon_0$ we have

$$M(T_1, Q_\varepsilon(a)) \le \fint_{Q_\varepsilon(a)} \fint_{Q_\varepsilon(a)} |g_m(y) - g_m(z)|\, dy dz \tag{101}$$

for some $m \le M_1(\varepsilon)$.

If $y, z \in Q_\varepsilon(a)$, we have

$$|f_m(y) - f_m(z)| \le |y - z| \|f_m\|_{\mathrm{Lip}} \le (N_m)^n \sqrt{n}\, \varepsilon \le C$$

(by (45)). Hence

$$|g_m(y) - g_m(z)| \le C, \tag{102}$$

since

$$|I(t) - I(s)| \le |t - s| + 1, \quad \forall\, t, s.$$

Combining (101) and (102) yields $M(T_1, Q_\varepsilon(a)) \le C$ and therefore

$$[T_1]_\varepsilon \le C, \quad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{103}$$

For $\varepsilon \in [\varepsilon_0, 1)$, we use (41) to assert that

$$[T_1]_\varepsilon \le \frac{2}{\varepsilon_0} \|T_1\|_{L^1(Q)} \le \frac{2}{\varepsilon_0} \|g\|_{L^1(Q)}. \tag{104}$$

Combining (103) with (104) we deduce that

$$[T_1]_\varepsilon \le C, \quad \forall\, \varepsilon \in (0, 1). \tag{105}$$

Estimate of $[T_2]_\varepsilon$. We claim that

$$\int_Q |\nabla g_m| \le C, \quad \forall\, m \ge 1, \tag{106}$$

and this implies via (9) that

$$[g_m]_\varepsilon \le C, \quad \forall\, m \ge 1,\ \forall\, \varepsilon \in (0, 1),$$

so that

$$[T_2]_\varepsilon \le C(M_2 - M_1) \le C, \quad \forall\, \varepsilon \in (0, 1) \tag{107}$$

(by (53)).

In order to prove (106), note that

$$\int_Q |\nabla g_m| = \sum_{k=1}^{(N_m)^{n-1}} \mathscr{H}^{n-1}([f_m = k]) = C \sum_{k=1}^{(N_m)^{n-1}-1} \left(1 - \frac{k}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}}$$
$$= C \sum_{\ell=1}^{(N_m)^{n-1}-1} \left(\frac{\ell}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}} \le C.$$

Estimate of $[T_3]_\varepsilon$. The technique for estimating $[S_3]_\varepsilon$ in the proof of Proposition 3 gives

$$[T_3]_\varepsilon \le C, \quad \forall\, \varepsilon \in (0,1). \tag{108}$$

Combining (105), (107) and (108) yields $g \in B$. □

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